CLASSIFICATION OF ALL POISSON-LIE STRUCTURES ON AN INFINITE-DIMENSIONAL JET GROUP

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Abstract. A local classification of all Poisson-Lie structures on an infinite-dimensional group G_∞ of formal power series is given. All Lie bialgebra structures on the Lie algebra $\mathcal{G}_\infty$ of G_∞ are also classified.

Let G_∞ be the group of formal power series in one variable $\{x(u) = \sum_{i=1}^\infty x_i u^i \mid x_1 \neq 0\}$, with a group multiplication $G_\infty \times G_\infty \to G_\infty$ being the substitution:

$$(xy)(u) := x\big(y(u)\big), \qquad \text{or} \qquad u \mapsto x(u), \tag{1}$$

and with an identity e the identity map $u \mapsto u$. The group G_∞ is the group of formal diffeomrphisms of $\mathbb{R}^1$ which leave the origin fixed. It is a projective limit $G_\infty = \varprojlim_n G_n$, where G_n are the finite-dimensional Lie groups of n-jets of the line at the origin. The multiplication in G_n is again defined by the substitution (1): $(\mathcal{X}_n \mathcal{Y}_n)(u) := \mathcal{X}_n(\mathcal{Y}_n(u)) \mod u^{n+1}$, where $\mathcal{X}_n(u)$ and $\mathcal{Y}_n(u)$ are polynomials in u of degree n. We define the space of smooth functions $C^\infty(G_\infty)$ to be the inductive limit $C^\infty(G_\infty) = \varinjlim_n C^\infty(G_n)$ of the spaces of smooth functions on the finite-dimensional groups G_n.

Following [1] we consider a multiplicative Poisson (Poisson-Lie) structure on G_∞ to be the bilinear skew-symmetric map $\{\ ,\ \}\colon C^\infty(G_\infty) \times C^\infty(G_\infty) \to C^\infty(G_\infty)$ defined by

$$\{f, g\} = \omega_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \tag{2}$$

for any $f, g \in C^\infty(G_\infty)$, such that the multiplication map $G_\infty \times G_\infty \to G_\infty$ is a Poisson map. Here $\omega_{ij} \in C^\infty(G_\infty)$, for any $i, j \in \mathbb{N}$, and a summation is assumed over repeated indices. The Poisson structure on $G_\infty \times G_\infty$ is taken to be the product Poisson structure. Note that the sum in (2) is finite since by definition f and g are functions of finite number of variables. Then the Jacobi identity for $\{\ ,\ \}$ implies that ω_{ij}'s satisfy

$$\omega_{ij} \frac{\partial \omega_{kl}}{\partial x_i} + \omega_{ik} \frac{\partial \omega_{lj}}{\partial x_i} + \omega_{il} \frac{\partial \omega_{jk}}{\partial x_i} = 0, \tag{3a}$$

for any $j, k, l \in \mathbb{N}$. The multiplicativity of the Poisson brackets (2) ($\{\ ,\ \}$ being a 1-cocycle) means that ω_{ij}'s must satisfy the following infinite system of functional

equations

$$\omega_{ij}(xy) = \omega_{kl}(x)\frac{\partial z_i}{\partial x_k}\frac{\partial z_j}{\partial x_l} + \omega_{kl}(y)\frac{\partial z_i}{\partial y_k}\frac{\partial z_j}{\partial y_l}, \qquad i, j \in \mathbb{N}, \tag{3b}$$

where $z = xy$. Note again that the sums in the right hand side of (3b) are finite. This is immediately seen from the explicit formulae

$$z_k = \sum_{i=1}^{k} x_i \sum_{(\sum_{\alpha=1}^{i} j_\alpha)=k} y_{j_1} \ldots y_{j_i}, \qquad k \geq 1,$$

for the coordinates of z. From (3b) also follows that $\omega_{ij}(e) = 0$.

Do such structures exist on G_∞? It is by no means obvious that such structures exist. For example:
(i) Let us consider the 3-dimensional factor group $G_3 = G_\infty \ (mod) \ u^n$, for $n \geq 4$. Then there exists a Poisson-Lie structure on G_3 described by

$$\begin{aligned}
\{x_1, x_2\} &= x_1 x_2 \\
\{x_1, x_3\} &= 4x_2^2 - 2x_1 x_3 \\
\{x_2, x_3\} &= 6\frac{x_2^3}{x_1} - 5x_2 x_3,
\end{aligned}$$

where x_1, x_2, x_3 are the coordinate functions on the group G_3. However, this Poisson-Lie structure can not be extended to a Poisson-Lie structure on G_∞.
(ii) We conjecture that there are no non-trivial Poisson-Lie structures on the group of diffeomorphisms of S^1.
Also a second question arises: If such structures exist, could they be classified? The answer to the first question is given by the following theorem.

Theorem 1. *For every natural number $d \in \mathbb{N}$, and every sequence $M_d = (\mu_n)_{n=1}^{\infty}$, such that $\mu_n = 0$ for $1 \leq n \leq d$ and $\mu_{d+1} \neq 0$, one has the following infinite-*

parameter family of Poisson-Lie structures on G_∞,

$$(4)\quad \omega_{ij}(x) = \sum_{p=1}^{i}\sum_{q=1}^{j} p x_p q x_q \lambda_{i-p+1,j-q+1} - \\ - \sum_{p=1}^{i}\sum_{q=1}^{j} \lambda_{pq} \sum_{\sum_{k=1}^{p} r_k = i} x_{r_1}\dots x_{r_p} \sum_{\sum_{l=1}^{q} s_l = j} x_{s_1}\dots x_{s_q},$$

where

$$(5)\qquad \lambda_{mn} = \frac{1}{\mu_{d+1}}\left[\mu_m\lambda_{d+1,n} - \mu_n\lambda_{d+1,m}\right] \qquad \forall\ m,n \geq 1.$$

Here, $\lambda_{d+1,n}$ are given by rational functions $\lambda_{d+1,n} = \lambda_{d+1,n}(\mu_{d+1},\dots,\mu_{d+n})$ for $n \geq 1$, which are computed by the following recursive formula

$$(6)\quad \lambda_{d+1,n} = -\frac{1}{(d-n+1)\mu_{d+1}}\left[d\mu_{d+1}\mu_{n+d} - \sum_{s=1}^{n-1}(n+d-2s+1)\mu_{n+d-s+1}\lambda_{s,d+1}\right],$$

where $\lambda_{1,d+1} = \mu_{d+1}$, and there exists the following single relation between the μ_n's (with $n \geq d+1$)

$$(7)\qquad \mu_{2d+1} = -\frac{1}{d\mu_{d+1}}\sum_{s=2}^{d} 2(d+1-s)\mu_{2d+2-s}\lambda_{s,d+1},$$

which are otherwise subject to no other restrictions. (We implicitly assume that $\lambda_{mn} = 0$ whenever $m < 1$ or $n < 1$.)

The classification is given by Corollary 2 below.

Remark. The relation (7) follows from (6) when $n = d+1$, and this is the only value of n for which the expression $(d-n+1)\lambda_{d+1,n}\mu_{d+1}$ equals zero.

The proof of Theorem 1 is rather technical [5,6]. We confine ourselves to give here the main ideas and tools used. Let $\mathcal{V} = \{u, v, w, \dots\}$ be a countable

set and let $C^\infty(G_\infty)[[\mathcal{V}]]$ be the ring of formal power series generated by $\mathcal{V}$ over $C^\infty(G_\infty)$. Let X_i, $i \in \mathbb{N}$, be the coordinate functions on G_∞. That is, $X_i(x) = x_i$ for $x \in G_\infty$. Introduce the formal series $X(u) := \sum_{i,j=1}^\infty X_i u^i$. Then $x(u) = X(u)(x) = \sum_{i,j=1}^\infty x_i u^i$, and $\omega_{ij} = \{X_i, X_j\}$. Define the formal series $\Omega(u, v; X) := \sum_{i,j=1}^\infty \omega_{ij} u^i v^j$. Thus $\Omega(u, v; X)$ is a generating series for the brackets ω_{ij}. After evaluation at $x \in G_\infty$ one has $\Omega(u, v; x) = \sum_{i,j=1}^\infty \omega_{ij}(x) u^i v^j$. The multiplicativity (3b) of the Poisson brackets on G_∞ is equivalent to $\Omega(u, v; x)$ satisfying the following functional equation

$$\Omega(u, v; xy) = \Omega(y(u), y(v); x) + \Omega(u, v; y)x'(y(u))x'(y(v)). \tag{8}$$

Here x' denotes the derivative of x with respect to its argument. The general solution of (8) is given by

$$\Omega(u, v; x) = \varphi(u, v)x'(u)x'(v) - \varphi(x(u), x(v)), \tag{9}$$

where $\varphi(u, v)$ is a formal series in u, v subject to the conditions:
(i) $\varphi(u, v)$ is divisible by u and v,
(ii) $\varphi(u, v) = -\varphi(v, u)$.
The map $\{\ ,\ \}\colon C^\infty(G_\infty)\times C^\infty(G_\infty) \to C^\infty(G_\infty)$ induces a map $\{\ ,\ \}\colon C^\infty(G_\infty)[[\mathcal{V}]]\times C^\infty(G_\infty)[[\mathcal{V}]] \to C^\infty(G_\infty)[[\mathcal{V}]]$. In particular one has

$$\{X(u), X(v)\} = \sum_{i,j=1}^\infty \{X_i, X_j\} u^i v^j = \Omega(u, v; X).$$

Then the Jacobi identities (3a) are equivalent to the single equation

$$\{X(w), \{X(u), X(v)\}\} + \{X(u), \{X(v), X(w)\}\} + \{X(v), \{X(w), X(u)\}\} = 0,$$

which, after a short calculation using the explicit formula (9), implies that $\varphi(u, v)$ must satisfy the following functional partial differential equation

$$\varphi(u, v)\big[\partial_u \varphi(w, u) + \partial_v \varphi(w, v)\big] + c.p. = 0. \tag{10}$$

Thus the content of Theorem 1 is a description of all solutions of (10) satisfying (i) and (ii). The relation between (9) and (4) is $\varphi(u,v) = \sum_{i,j=1}^{\infty} \lambda_{ij} u^i v^j$. The solution (4) completely describes the space of solutions of (10). An equivalent description can be given as follows.

Theorem 1a. *For each $d \in \mathbb{N}$, and any (formal series) $f_d(u), g_d(u)$ such that $f_d'(u)g_d(u) - f_d(u)g_d'(u) = -d\mu_{d+1} f_d(u)$, where $\mu_{d+1} \neq 0$ is an arbitrary parameter, and f_d has a zero of order $d+1$ at $u=0$, there is a solution of* (10) *given by*

$$\varphi_d(u,v) = \frac{1}{\mu_{d+1}} \Big[f_d(u) g_d(v) - f_d(v) g_d(u) \Big].$$

The set of all solutions of (10) *is described in this way.*

Corollary 1. *A subclass of the above family is the following countable family of Poisson-Lie structures. For each $d \in \mathbb{N}$, choosing $M_d = (0,1,0,0,\ldots)$, one has*

$$(11) \quad \omega_{ij}(x) = (i-d) j x_j x_{i-d} - i(j-d) x_i x_{j-d} + \\ + x_i \sum_{\sum_{k=1}^{d+1} s_k = j} x_{s_1} \ldots x_{s_{d+1}} - x_j \sum_{\sum_{k=1}^{d+1} s_k = i} x_{s_1} \ldots x_{s_{d+1}},$$

for every $i, j \geq 1$. (We adopt the convention that $x_i = 0$ whenever $i < 1$).

This family corresponds to the set of solutions $\varphi(u,v) = uv(u^d - v^d)$, $d \in \mathbb{N}$, of (10).

Let $i_\infty : G_\infty \to G_\infty$ be the inversion map defined by $i_\infty(x) = x^{-1}$ for every $x \in G_\infty$.

Theorem 2. *The map $i_\infty : G_\infty \to G_\infty$ is an anti-Poisson map.*

In other words one has $\{f,g\}(i_\infty(x)) = -\{f,g\}(x)$, for every $f, g \in C^\infty(G_\infty)$ and $x \in G_\infty$. The proof uses only the explicit form (9) of the brackets on G_∞. Let

$\overline{X}(u)$ be the inverse of $X(u)$. Then one has $\overline{X}(X(u)) = u$, and $X(\overline{X}(u)) = u$, as well as $\overline{X}'(X(u))X'(u) = 1$, and $X'(\overline{\mathcal{X}}(u))\overline{X}'(u) = 1$. On the other hand

$$
\begin{aligned}
0 &= \{u, X(v)\} \\
&= \{\overline{X}(X(u)), X(v)\} \\
&= \{\overline{X}(w), X(v)\}|_{w=X(u)} + \overline{X}'(w)|_{w=X(u)}\{X(u), X(v)\}.
\end{aligned}
$$

Therefore,

$$
\{X(v), \overline{X}(w)\}|_{w=X(u)} = \overline{X}'(w)|_{w=X(u)}\{X(u), X(v)\}. \tag{12}
$$

Also, one has the chain of identities

$$
\begin{aligned}
0 &= \{v, \overline{X}(w)\}|_{w=X(u)} \\
&= \{\overline{X}(X(v)), \overline{X}(w)\}|_{w=X(u)} \\
&= \{\overline{X}(s), \overline{X}(w)\}|_{s=X(v),w=X(u)} + \overline{X}'(s)|_{s=X(v)}\{X(v), \overline{X}(w)\}|_{w=X(u)}.
\end{aligned}
$$

Using (9) and (12), one rewrites the last identity as

$$
\begin{aligned}
0 &= \varphi(X(v), X(u))\overline{X}'(X(v))\overline{X}'(X(u)) - \varphi(v, u) \\
&\qquad + \overline{X}'(X(v))\overline{X}'(X(u))\left[\varphi(u, v)X'(u)X'(v) - \varphi(X(u), X(v))\right] \\
&= \{\overline{X}(s), \overline{X}(w)\}|_{s=X(v),w=X(u)} + \varphi(u, v) - \overline{X}'(w)\overline{X}'(s)\varphi(w, s).
\end{aligned}
$$

Thus,

$$
\{\overline{X}(w), \overline{X}(s)\} = -\left[\overline{X}'(w)\overline{X}'(s)\varphi(w, s) - \varphi(\overline{X}(w), \overline{X}(s))\right],
$$

and this concludes the proof.

Remark. In the beginning of the theory of Poisson-Lie groups, the property of the inversion map $i : G \to G$ to be anti-Poisson was considered as an axiom [1,4]. However, for finite-dimensional groups this property can be deduced from the other axioms [5,6]. For infinite-dimensional groups such deduction is not likely.

To show that formula (4) provides *all* Poisson-Lie structures on G_∞, we now turn to the Lie algebra $\mathcal{G}_\infty$ of G_∞. Let $\{e_n\}_{n\geq 0}$ be a basis of $\mathcal{G}_\infty$, and let α be a 1-cochain $\alpha\colon \mathcal{G}_\infty \to \mathcal{G}_\infty \hat{\wedge} \mathcal{G}_\infty$, which we write in the above basis as $\alpha(e_n) = \sum_{i,j=0}^{\infty} \alpha_n^{ij} e_i \wedge e_j$, where α takes values in the completed tensor product $\mathcal{G}_\infty \widehat{\otimes} \mathcal{G}_\infty = \bigoplus_{n=1}^{\infty} \Big(\oplus_{i+j=n} \mathcal{G}_i \otimes \mathcal{G}_j \Big)$, where each $\mathcal{G}_i$ is a one-dimensional subspace of $\mathcal{G}_\infty$ spanned by e_i. The Lie algebra structure on $\mathcal{G}_\infty$ is given by

$$\big[e_n, e_m\big] = (n-m) e_{n+m} \qquad \forall\ n, m \geq 0.$$

Then the map α equips $\mathcal{G}_\infty$ with a Lie bialgebra structure [1] iff

$$(i) \qquad \tau \circ \alpha = -\alpha$$

$$(ii) \qquad \alpha\big([a,b]\big) = a.\alpha(b) - b.\alpha(a), \quad a, b \in \mathcal{G}_\infty,$$

$$(iii) \qquad [1 \otimes 1 \otimes 1 + (\tau \otimes 1)(1 \otimes \tau) + (1 \otimes \tau)(\tau \otimes 1)](1 \otimes \alpha) \circ \alpha = 0,$$

where τ is the transposition map $\tau\colon \mathcal{G}_\infty \widehat{\otimes} \mathcal{G}_\infty \to \mathcal{G}_\infty \widehat{\otimes} \mathcal{G}_\infty$ defined by $\tau(a \otimes b) = b \otimes a$, for any $a, b \in \mathcal{G}_\infty$, and the dot stands for the action of $\mathcal{G}_\infty$ on $\mathcal{G}_\infty \widehat{\wedge} \mathcal{G}_\infty$ induced by the adjoint action of $\mathcal{G}_\infty$ on itself. In the case when α is a 1-coboundary one has $\alpha(a) = a.r$, where $r \in \mathcal{G}_\infty \hat{\wedge} \mathcal{G}_\infty$ is a 0-cochain referred to as the classical r-matrix [1,3]. In the latter case, (iii) above is equivalent to $a. < r, r >= 0$, for any $a \in \mathcal{G}_\infty$. Here $< r, r >:= \big[r^{12}, r^{13}\big] + \big[r^{12}, r^{23}\big] + \big[r^{13}, r^{23}\big]$, and $\big[r^{12}, r^{13}\big] := \sum_{i,j,k,l=0}^{\infty} r^{ij} r^{kl} [e_i, e_k] \wedge e_j \wedge e_l$, etc., where $r = \sum_{i,j=0}^{\infty} r^{ij} e_i \wedge e_j$.

Theorem 3. *The first cohomology group* $H^1(\mathcal{G}_\infty, \mathcal{G}_\infty \hat{\wedge} \mathcal{G}_\infty) = 0$. *That is, all* 1*-cocycles* $\alpha\colon \mathcal{G}_\infty \to \mathcal{G}_\infty \hat{\wedge} \mathcal{G}_\infty$ *are coboundaries.*

This can be proven by analyzing the infinite system of linear equations

$$\alpha\big([e_n, e_m]\big) = e_n.\alpha(e_m) - e_m.\alpha(e_n), \qquad n, m \geq 0,$$

using its symmetries, and inductive arguments. Then an analysis of the system of equations $e_n. < r, r >= 0$, $n \geq 0$, shows that it is equivalent to $< r, r >= 0$ (CYBE,

the classical Yang-Baxter equation). This turned out to be a specific property of the algebra $\mathcal{G}_\infty$ [5,6]. Thus, all Lie bialgebra structures on $\mathcal{G}_\infty$ are given by solutions of the classical Yang-Baxter equation. Moreover the following theorem holds.

Theorem 4. *There is a one-to-one correspondence between the coboundary Lie bialgebra structures on $\mathcal{G}_\infty$ and the Poisson-Lie structures* (4) *on G_∞, the correspondence being given by $r^{ij} = \lambda_{i+1,j+1}$, for every $i, j \geq 0$. Thus, for each $d \in \mathbb{N}$ we have the following infinite-parameter family of Lie bialgebra structures on $\mathcal{G}_\infty$ given by $\alpha(e_n) = \sum_{i,j=0}^{\infty} \alpha_n^{ij} e_i \wedge e_j$, where*

$$\begin{aligned}\alpha_n^{ij} &= (2n-i)r^{i-n,j} + (2n-j)r^{i,j-n} \\ &= (2n-i)\lambda_{i-n+1,j+1} + (2n-j)\lambda_{i+1,j-n+1}, \qquad \forall\, n, i, j \geq 0,\end{aligned}$$

and λ_{nm} are subject to the same conditions as described in Theorem 1.

Corollary 2. *Thus, Theorem* 1 *describes all Poisson-Lie structures on the group G_∞.*

The proof of Theorem 4 consists of showing that each Lie bialgebra structure on $\mathcal{G}_\infty$ can be integrated to a unique Poisson-Lie structure on the group G_∞. To show this one has to show that the following infinite system of linear partial differential equations

$$\begin{aligned}\sum_{i=j}^{n}(i+1-j)x_{i+1-j}\frac{\partial \omega_{mn}}{\partial x_i} &= \omega_{m+1-j,n}(x)(m+1-j) + \omega_{m,n+1-j}(x)(n+1-j) + \\ &+ \sum_{k=1}^{m}\sum_{l=1}^{n}\alpha_j^{kl}(m+1-k)(n+1-l)x_{m+1-k}x_{n+1-l},\end{aligned}$$

where $1 \leq j \leq n$, and $m, n \in \mathbb{N}$, has a unique solution. Here α_n^{ij} are the coalgebra structure constants of $\mathcal{G}_\infty$. The above system can be obtained by differentiating (3b) with respect to y and setting $y = e$, in which case $\alpha_n^{ij} = \frac{\partial \omega_{ij}}{\partial y_n}\big|_{y=e}$. The

existence of a solution is furnished by Theorem 1 since any solution of (3b) is a solution of the above system. To show that it is unique one shows inductively that the corresponding homogeneous system

$$\sum_{i=j}^{n}(i+1-j)x_{i+1-j}\frac{\partial \omega_{mn}}{\partial x_i} = \omega_{m+1-j,n}(x)(m+1-j)+\omega_{m,n+1-j}(x)(n+1-j)$$

has only the trivial solution.

A subfamily of Lie bialgebra structures that corresponds to the family of Poisson-Lie structures (11) is given by

$$\alpha_d(e_n) = 2ne_d \wedge e_n - 2(n-d)e_0 \wedge e_{d+n}, \tag{13}$$

for each $d \in \mathbb{N}$. The entries of the r-matrix in this case are $r^{ij} = \delta^1_{i+1}\delta^{d+1}_{j+1} - \delta^{d+1}_{i+1}\delta^1_{j+1} = \lambda_{i+1,j+1}$. The family (13) of Lie bialgebra structures on $\mathcal{G}_\infty$ had been found and studied in [2,7]. Also, we describe below a 1-parameter family, $\alpha_{d,\lambda}$, of Lie bialgebra structures, for each $d \geq 1$, of which the family (13) is a subfamily obtained after the specialization $\lambda = 0$. Namely,

$$\begin{aligned}\alpha_{d,\lambda}(e_n) =& 2\sum_{i=d+n}^{\infty}(2n-i)\lambda^{i-(n+d)}(d-1)^{i-(n+d)}e_0\wedge e_i - 2n\sum_{i=d}^{\infty}\lambda^{i-d}(d-1)^{i-d}e_i\wedge e_n \\ &+ 2\sum_{i=d+n}^{\infty}\sum_{j=1}^{d-1}(2n-i)\lambda^{i+j-(n+d)}(d-1)^{i+j-(n+d+1)}e_i\wedge e_j \\ &+ 2\sum_{i=d}^{\infty}\sum_{j=n+1}^{d+n-1}(2n-j)\lambda^{i+j-(n+d)}(d-1)^{i+j-(n+d+1)}e_i\wedge e_j.\end{aligned}$$

Again, the right-hand-side of the above formula is an element of the completed tensor product $\mathcal{G}_\infty\widehat{\otimes}\mathcal{G}_\infty$. This family corresponds to the following solution of (10):

$$\varphi_{d,\lambda}(u,v) = \frac{1}{[1-(d-1)\lambda u][1-(d-1)\lambda v]}\left\{uv(v^d-u^d)+\lambda d u^2v^2(u^{d-1}-v^{d-1})\right\}.$$

We conclude by noting that as a consequence of Theorem 4 the equation (10) is a functional realization of the classical Yang-Baxter equation for $\mathcal{G}_\infty$.

The complete proofs of the above results will be published elsewhere [5,6].

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